

# LIFTING REPRESENTATIONS OF $\mathbb{Z}$ -GROUPS

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**ABSTRACT:** Let  $K$  be the kernel of an epimorphism  $G \rightarrow \mathbb{Z}$ , where  $G$  is a finitely presented group. If  $K$  has infinitely many subgroups of index 2, 3 or 4, then it has uncountably many. Moreover, if  $K$  is the commutator subgroup of a classical knot group  $G$ , then any homomorphism from  $K$  onto the symmetric group  $S_2$  (resp.  $\mathbb{Z}_3$ ) lifts to a homomorphism onto  $S_3$  (resp. alternating group  $A_4$ ).

**1. Introduction.** Let  $G$  be a finitely presented group with infinite abelianization. Given an epimorphism  $\chi : G \rightarrow \mathbb{Z}$ , we denote its kernel by  $K$ . Examples of special interest arise in knot theory; if  $G$  is the group  $\pi_1(S^3 \setminus k)$  of a knot  $k \subset S^3$  and  $\chi$  is abelianization, then  $K$  is the commutator subgroup of  $G$ .

In general,  $K$  need not be finitely generated. Nevertheless, the Reidemeister-Schreier method [LS77] ensures that it has a group presentation composed of finitely many families of generators  $a_j, b_j, \dots, c_j$  ( $j \in \mathbb{Z}$ ) and relators  $r_k, s_k, \dots, t_k$  ( $k \in \mathbb{Z}$ ) such that any relator in a family can be gotten from any other by shifting all of the indices of its generators by a constant. (Conversely, any group  $K$  with such a presentation arises as a kernel  $\chi : G \rightarrow \mathbb{Z}$  for some finitely presented  $G$ .) Clearly,  $K$  admits a nontrivial  $\mathbb{Z}$ -action by automorphisms. The action is the restriction to  $K$  of conjugation in  $G$  by a preimage  $x \in \chi^{-1}(1)$ ; actions corresponding to different preimages are related by an inner automorphism of  $K$ . For this reason we call  $K$  a *finitely presented  $\mathbb{Z}$ -group* (cf. [Ro96]).

In [SW96] the authors exploited this special structure, showing that for any finite group  $\Sigma$ , the set of representations  $\text{Hom}(K, \Sigma)$  has the structure of a *shift of finite type*, a compact 0-dimensional dynamical system completely described by a finite directed graph  $\Gamma$ ; in particular, there is a bijection between  $\text{Hom}(K, \Sigma)$  and bi-infinite paths in  $\Gamma$ . Techniques of symbolic dynamics can be used to understand  $\text{Hom}(K, \Sigma)$ . Details are reviewed in §2.

Given any group  $K$ , its subgroups of index no greater than  $r$  are in finite-to-one correspondence with representations  $\rho : K \rightarrow S_r$ , where  $S_r$  is the symmetric group on  $\{1, \dots, r\}$ . The correspondence can be described by  $\rho \mapsto \{g \in K \mid \rho(g)(1) = 1\}$ . The preimage of a subgroup of index exactly  $r$  consists of  $(r-1)!$  transitive representations. By a *transitive* representation we mean a representation  $\rho$  such that  $\rho(K)$  operates transitively on  $\{1, \dots, r\}$ . Note that  $K$  contains finitely (resp. countably, uncountably) many subgroups of index  $r$  if and only if  $\text{Hom}(K, S_r)$  contains finitely (resp. countably, uncountably) many transitive representations.

In [SW99] we applied techniques of symbolic dynamics to study  $\text{Hom}(K, S_r)$ . Under the hypothesis that  $K$  contains an abelian HNN base for  $G$  (see §2) we proved that if  $K$  contains infinitely many subgroups of some finite index  $r$ , then it contains uncountably many. Our first result is that this dichotomy continues to hold even if the hypothesis is removed, provided that  $r < 5$ .

**Theorem 3.4. (Dichotomy)** Let  $K$  be a finitely presented  $\mathbb{Z}$ -group. If  $K$  contains infinitely many subgroups of index 2, 3 or 4, then  $K$  contains uncountably many subgroups of that index.

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The conclusion does not hold if  $r > 4$  (see Example 3.4.) As an immediate consequence of the above theorem and Corollary 1.3 of [SW99] we obtain

**Corollary 3.5.** Let  $K$  be a finitely presented  $\mathbb{Z}$ -group. If  $K$  contains infinitely many subgroups of index  $r = 2, 3$  or  $4$ , then  $K$  contains uncountably many subgroups of any index greater than or equal to  $r!$ .

When  $G$  is a knot group, topology imposes restrictions on the structure of its commutator subgroup  $K$ .

**Theorem 4.3** Assume that  $K$  is the commutator subgroup of a knot group. Then (i) any representation from  $K$  onto  $S_2$  lifts to a representation onto  $S_3$ ; (ii) any representation from  $K$  onto  $\mathbb{Z}_3$  lifts to a representation onto the alternating group  $A_4$ .

The conclusions of Theorem 4.3 do not hold for arbitrary  $\mathbb{Z}$ -groups (see Example 4.4).

The commutator subgroup of a nontrivial fibered knot is free of rank at least two. Such a group maps onto any symmetric group (and consequently contains subgroups of every index). Does such a conclusion hold for the commutator subgroup of any *nonfibered* knot? We offer a partial answer (see Corollary 3.9).

Much of this paper was inspired by C. Livingston's paper [Li95]. In it he revisits classical results about lifting knot group representations from the perspective of obstruction theory. We found that with the aid of symbolic dynamics many of the techniques extend in a natural way to knot commutator subgroups. From this approach we obtain new insights into the structure of  $\text{Hom}(K, \Sigma)$ .

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**2. Symbolic dynamics and representation shifts.** Assume that  $G$  is a finitely presented group with epimorphism  $\chi : G \rightarrow \mathbb{Z}$ . Then  $G$  can be described as an *HNN extension*  $\langle x, B \mid x^{-1}ax = \phi(a), \forall a \in U \rangle$ . Here  $x \in \chi^{-1}(1)$ , while  $B$  is a finitely generated subgroup of  $K = \ker \chi$ . The map  $\phi$  is an isomorphism between finitely generated subgroups  $U, V$  of  $B$ . The subgroup  $B$  is an *HNN base*,  $x$  is a *stable letter*,  $\phi$  is an *amalgamating map*. Details can be found in [LS77]. It is possible to choose  $B$  so that it contains any prescribed finite subset of  $K$  (see [Si96], for example).

Conjugation by  $x$  induces an automorphism of  $K$ . Letting  $B_j = x^{-j}Bx^j$ ,  $U_j = x^{-j}Ux^j$  and  $V_j = x^{-j}Vx^j$ ,  $j \in \mathbb{Z}$ , we can express  $K$  as an infinite free product in which each subgroup  $V_j$  is amalgamated with  $U_{j+1}$ :

$$K = \langle B_j \mid x^{-j}\phi(u)x^j = x^{-j-1}ux^{j+1}, \forall u \in U, \forall j \in \mathbb{Z} \rangle.$$

When  $\Sigma$  is a finite group, the set  $\text{Hom}(K, \Sigma)$  can be described by finite directed graph  $\Gamma$ . The vertex set consists of all representations  $\rho_0 : U \rightarrow \Sigma$ , a set that is finite since  $U$  is finitely generated. If  $\bar{\rho}_0$  is a representation from  $B$  to  $\Sigma$ , then we draw a directed edge labeled by  $\bar{\rho}_0$  from the vertex  $\rho_0 = \bar{\rho}_0|_U$  to the vertex  $\rho'_0 = \bar{\rho}_0|_V \circ \phi$ . Consider any bi-infinite path in  $\Gamma$  given by an edge sequence

$$\cdots \bar{\rho}_{-2} \bar{\rho}_{-1} \bar{\rho}_0 \bar{\rho}_1 \bar{\rho}_2 \cdots.$$

The representations from  $B_j$  to  $\Sigma$  defined by  $y \mapsto \bar{\rho}_j(x^j y x^{-j})$  have a unique common extension  $\rho : K \rightarrow \Sigma$ . In this way bi-infinite paths of  $\Gamma$  correspond to elements of  $\text{Hom}(K, \Sigma)$ .

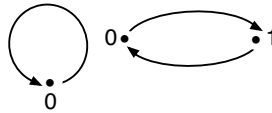
Let  $E$  be a finite set (with discrete topology) and give  $E^{\mathbb{Z}}$  the product topology. We define the *shift* homeomorphism  $\sigma$  by  $(\sigma y)_j = y_{j+1}$  for  $y = (y_j) \in E^{\mathbb{Z}}$ . The dynamical system  $(E^{\mathbb{Z}}, \sigma)$  is called the *full shift* on the symbol set  $E$ . Any closed  $\sigma$ -invariant subset is a *subshift*. In a slight abuse of notation, we will use the same symbol  $\sigma$  for the restriction of  $\sigma$  to a subshift, and also for subshifts on different symbol sets. A subshift  $Y$  is an  $n$ -step *shift of finite type* if there is a subset  $S$  of  $E^{n+1}$ , the set of *allowed  $(n+1)$ -blocks*, such that  $Y$  consists of precisely those  $y = (y_j)$  for which  $y_j \cdots y_{j+n} \in S$ , for all  $j$ . (Following the custom in symbolic dynamics and information theory, we write elements of  $E^n$  as words  $y_1 \cdots y_n$  instead of  $n$ -tuples.) A finite directed graph  $\Gamma$  determines a 1-step shift of finite type  $X_\Gamma$ , the *edge shift* of  $\Gamma$ : the symbol set is the edge set  $E$  and  $ee'$  is an allowed 2-block if the terminal vertex of  $e$  is the initial vertex of  $e'$ . Thus  $X_\Gamma$  is the set of elements of  $E^{\mathbb{Z}}$  that correspond to bi-infinite paths in  $\Gamma$ . The graph constructed in the preceding paragraph presents  $\text{Hom}(K, \Sigma)$  as a shift of finite type. As we showed in [SW96], the product topology on the shift space coincides with the compact-open topology on  $\text{Hom}(K, \Sigma)$  and the shift map  $\sigma$  is induced by the map  $\sigma_x : \text{Hom}(K, \Sigma) \rightarrow \text{Hom}(K, \Sigma)$  given by

$$(\sigma_x \rho)(a) = \rho(x^{-1}ax), \quad \forall a \in K.$$

Two dynamical systems  $(Y, \tau)$  and  $(Y', \tau')$  are *topologically conjugate*, or simply *conjugate*, if there is a homeomorphism  $h : Y \rightarrow Y'$  with  $\tau' \circ h = \tau$ . Although the graph presenting  $\text{Hom}(K, \Sigma)$  depends on the choice of HNN base  $B$  and stable letter  $x$ , the corresponding shift of finite type is uniquely determined up to conjugacy by  $G$  and  $\chi$ . As in [SW96] we call the pair  $(\text{Hom}(K, \Sigma), \sigma_x)$  the *representation shift* of  $K$  in  $\Sigma$ , denoted by  $(\Phi_\Sigma, \sigma)$  or more simply  $\Phi_\Sigma$ . An algorithm, based on the Reidemeister-Schreier method, for obtaining a graph presenting  $\Phi_\Sigma$  from a finite presentation of  $G$  is given in [SW96].

**Example 2.1.** Consider the Baumslag-Solitar group  $G = \langle x, a \mid ax = a^2x \rangle$  together with the epimorphism  $\chi : G \rightarrow \mathbb{Z}$  mapping  $x \mapsto 1$  and  $a \mapsto 0$ . The group  $G$  has an HNN decomposition such that  $B = U$  is the infinite cyclic subgroup  $\langle a \rangle$ ,  $V = \langle a^2 \rangle$  and  $\phi(a) = a^2$ . The kernel  $K$  of  $\chi$  has presentation  $\langle a_j \mid a_j^2 = a_{j+1} \forall j \in \mathbb{Z} \rangle$ .

In this example  $\Phi_{\mathbb{Z}_3}$  has exactly 3 elements: the trivial representation; the representation  $\rho$  mapping  $a_{2j} \mapsto 1$  and each  $a_{2j+1} \mapsto 2$ ; and the representation  $\sigma\rho$  mapping  $a_{2j} \mapsto 2$  and each  $a_{2j+1} \mapsto 1$ . The graph  $\Gamma$  that describes the representation shift appears in Figure 1. Here we label the vertex  $\rho_0$  by  $\rho_0(a)$ .



**Figure 1:** Graph  $\Gamma$  describing  $\Phi_{\mathbb{Z}_3}$

We close this section with some notions from symbolic dynamics that will be needed in the sequel. We refer the reader to [LM95] for additional background.

If  $Y$  is a subshift of  $E^{\mathbb{Z}}$  and  $n$  a positive integer, we may construct a conjugate subshift  $Y^{(n)}$  of  $(E^n)^{\mathbb{Z}}$ , the  $n$ -block presentation of  $Y$ , by sending the sequence  $(y_j) \in Y$  to the sequence of overlapping  $n$ -blocks

$$\dots (y_j \dots y_{j+n-1})(y_{j+1} \dots y_{j+n})(y_{j+2} \dots y_{j+n+1}) \dots$$

If  $Y$  is an (at most)  $n$ -step shift of finite type then  $Y^{(n)}$  is a 1-step shift of finite type described by a graph in which the edges are allowed  $(n+1)$ -blocks of  $Y$  and the initial and terminal vertices of an edge are its initial and terminal  $n$ -blocks. If we replace the HNN base  $B = B_0$  in our construction of a graph presenting  $\Phi_{\Sigma}$  by the larger base  $B^{(n)} = \langle B_0 \cup B_1 \cup \dots \cup B_{n-1} \rangle$ , we obtain a graph for the  $n$ -block presentation.

We may, and will, assume that the graph  $\Gamma$  of a shift of finite type  $X_{\Gamma}$  has been “pruned” to remove all vertices and edges that do not lie on a bi-infinite path. Then  $X_{\Gamma}$  is *irreducible* if  $\Gamma$  is strongly connected, that is, there is a path from every vertex to every other vertex. The *irreducible components* of  $X_{\Gamma}$  are the finite type subshifts corresponding to the maximal strongly connected subgraphs of  $\Gamma$ .

A point  $y$  of a subshift has *period*  $r$  if  $\sigma^r y = y$ . We will refer to periodic points of the representation shift  $\Phi_{\Sigma}$  as periodic representations. Periodic orbits of  $X_{\Gamma}$  correspond to closed paths in  $\Gamma$ . If a point  $y$  of a shift of finite type has period at most  $r$  then it is easily seen that its orbit is represented by a simple closed path in the graph of the  $r$ -block presentation. The periodic points of  $X_{\Gamma}$  form a dense subset if and only if  $X_{\Gamma}$  is the union of its irreducible components. The subshift  $X_{\Gamma}$  is finite if and only if  $\Gamma$  is a disjoint union of cycles, and uncountable if and only if  $\Gamma$  has a strongly connected subgraph containing more than one cycle.

A *Markov group* is a shift of finite type that is simultaneously a group under an operation that is preserved by the shift map. If  $E$  is a group then the full shift  $E^{\mathbb{Z}}$  is a Markov group under coordinatewise multiplication, and so is any finite type subshift of  $E^{\mathbb{Z}}$  that is also a subgroup. For example,  $\Phi_{\Sigma} = \text{Hom}(K, \Sigma)$  is a Markov group if  $\Sigma$  is abelian. Theorem 6.3.3 of [Ki98] describes the structure of a Markov group: The irreducible component that contains the identity element is topologically conjugate to a full shift (possibly the trivial shift  $\{e\}^{\mathbb{Z}}$ ), and the entire Markov group is conjugate to the product of this full shift with a finite Markov group (which may also be trivial). It follows that a Markov group has dense periodic point set. It is finite if the full shift factor is trivial, and otherwise every irreducible component is uncountable.

If  $K$  is the commutator subgroup of a knot group, then  $\Phi_{\Sigma}$  must be finite for every abelian  $\Sigma$ . For otherwise, the full shift component would contain a nontrivial fixed point  $\rho$ . Then  $\rho(a) = \sigma\rho(a) = \rho(x^{-1}ax)$ , or  $\rho(a^{-1}x^{-1}ax) = e$ , for all  $a \in \rho$ . But the knot group is the normal closure of  $x$ , and so the elements  $a^{-1}x^{-1}ax$  generate  $K$  (see [HK78], for example.)

**3. Lifting representations.** Assume that  $E$  is an extension of  $\Sigma$  by an abelian group  $A$ :

$$0 \rightarrow A \rightarrow E \xrightarrow{p} \Sigma \rightarrow 1.$$

For notational convenience, we identify  $A$  with its image  $i(A)$ , and regard the latter as a multiplicative group.

The group  $E$  acts on  $A$  by conjugation. Similarly, there is an induced action of  $\Sigma$  on  $A$  defined by  $a^y := \tilde{y}a\tilde{y}^{-1}$ , where  $\tilde{y}$  is any preimage of  $y$ ; in this way we regard  $A$  as a  $\Sigma$ -module. If a homomorphism  $\rho: K \rightarrow \Sigma$  lifts to  $\tilde{\rho}: K \rightarrow E$ , then  $A$  acts by conjugation on the set of liftings.

The homomorphism  $p$  induces a continuous mapping  $p^*: \Phi_E \rightarrow \Phi_{\Sigma}$ . Moreover,  $p^*$  is *shift-commuting* in the sense that  $p^* \circ \sigma = \sigma \circ p^*$ . If the extension splits then  $p^*$  is onto.

The following proposition is a standard application of the cohomology theory of group extensions (see [Br82], for example).

**Proposition 3.1.** Assume that  $\rho \in \Phi_\Sigma$  has a lifting  $\tilde{\rho} \in \Phi_E$ .

(i) The set of  $\rho$ -twisted cocycles,

$$C^1(K; \{A\}) = \{ \xi : K \rightarrow A \mid \xi(xy) = \xi(x)\xi(y)^{\rho(x)} \},$$

corresponds via  $\xi \mapsto \rho_\xi$ , where  $\rho_\xi(x) = \xi(x)\tilde{\rho}(x)$ , to the complete set of liftings;

(ii) Liftings  $\tilde{\rho}_1, \tilde{\rho}_2$  are  $A$ -conjugate if and only if  $\tilde{\rho}_2(\tilde{\rho}_1)^{-1}$  is a *coboundary*, a map of the form  $x \mapsto a^{\rho(x)}a^{-1}$ , for some fixed  $a \in A$ .

**Proof.**

(i) Given a derivation  $\xi$ , define  $\rho_\xi : K \rightarrow E$  by  $\rho_\xi(x) = \xi(x)\tilde{\rho}(x)$ . Then

$$\begin{aligned} \rho_\xi(xy) &= \xi(xy)\tilde{\rho}(xy) = \xi(x)\xi(y)^{\rho(x)}\tilde{\rho}(x)\tilde{\rho}(y) \\ &= \xi(x)\tilde{\rho}(x)\xi(y)\tilde{\rho}(y) = \rho_\xi(x)\rho_\xi(y). \end{aligned}$$

Hence  $\rho_\xi$  is in  $\Phi_E$ , and it is a lift of  $\rho$ .

Conversely, given a lift  $\hat{\rho}$  of  $\rho$ , define  $\xi : K \rightarrow A$  by  $\xi(x) = \hat{\rho}(x)(\tilde{\rho}(x))^{-1} \in \text{Ker}(p) = A$ . Then

$$\begin{aligned} \xi(xy) &= \hat{\rho}(xy)(\tilde{\rho}(xy))^{-1} = \hat{\rho}(x)\hat{\rho}(y)(\tilde{\rho}(y))^{-1}(\tilde{\rho}(x))^{-1} \\ &= \hat{\rho}(x)(\tilde{\rho}(x))^{-1}[\hat{\rho}(y)(\tilde{\rho}(y))^{-1}]^{\rho(x)} \\ &= \xi(x)\xi(y)^{\rho(x)}. \end{aligned}$$

Hence  $\xi \in C^1(K; \{A\})$  and  $\hat{\rho} = \rho_\xi$ . Clearly, distinct  $\xi$  give distinct  $\rho_\xi$ .

The proof of (ii) is equally routine, and we leave it to the reader. ■

**Remark 3.2.** Proposition 3.1 implies that the  $A$ -conjugacy classes of liftings correspond to elements of the cohomology group  $H^1(K; \{A\})$ , with coefficients in  $A$  twisted by the action of  $K$ . If  $X$  is a complex with  $\pi_1 X \cong K$ , then  $H^1(K; \{A\})$  is isomorphic to  $H^1(X, A)$  (see Proposition 2 of [Li95]).

**Lemma 3.3.** Let  $\rho \in \Phi_\Sigma$  be a periodic representation with lifting  $\tilde{\rho} \in \Phi_E$ . The preimage under  $p^*$  of the orbit of  $\rho$  is a shift of finite type with dense set of periodic points, that is, a disjoint union of irreducible shifts of finite type. The irreducible components are all finite or all uncountable.

**Proof.** As in §2, regard  $G$  as an HNN extension with finitely generated HNN base  $B$  and stable letter  $x \in \chi^{-1}(1)$ . Then  $\Phi_\Sigma$  is described by a graph with edge set  $\text{Hom}(B, \Sigma)$ , and  $\Phi_E$  by a graph with edge set  $\text{Hom}(B, E)$ . Let  $r$  be the least period of  $\rho$ . Replacing  $B$  with the larger HNN base  $B^{(r)} = \langle B_0 \cup \dots \cup B_{r-1} \rangle$  if necessary, we can assume that the orbit  $O$  of  $\rho$  is represented by a simple cycle in this graph. Then the set  $\tilde{O}$  of lifts of  $O$  is the finite type subshift of  $\Phi_E$  described by the subgraph consisting of the edges  $\eta \in \text{Hom}(B, E)$  for which  $p \circ \eta$  is an edge in the cycle presenting  $\rho$ .

For  $\xi \in C^1(K; \{A\})$  we define  $\sigma\xi : K \rightarrow A$  by  $\sigma\xi(y) = \xi(x^{-1}yx)$ . It is easy to check that  $\sigma\xi$  is a  $\sigma\rho$ -twisted cocycle. Since  $\rho$  has period  $r$ , we obtain an action of  $\tau = \sigma^r$  on  $C^1(K; \{A\})$ . We claim that

the pair  $(C^1(K; \{A\}), \tau)$  can be viewed as a shift of finite type. The symbol set is the set  $C^1(B^{(r)}; \{A\})$ . An element  $\xi \in C^1(K; \{A\})$  can be identified with the sequence of symbols  $\xi_j = \tau^j \xi|_{B^{(r)}}$ , and  $\xi\xi'$  is an allowed 2-block if  $\xi|_{V_{r-1}} \circ \phi = \xi'|_U$ .

It is straightforward to check that  $C^1(K, \{A\})$  is an abelian group under the operation  $(\xi + \eta)(y) = \xi(y)\eta(y)$ , and that  $\tau$  respects this addition. Hence  $(C^1(K, \{A\}), \tau)$  is a Markov group. By the Kitchens structure theorem cited in the preceding section, it is a disjoint union of finitely many shifts of finite type which are all finite or all uncountable.

By Proposition 3.1, the set  $\mathcal{P}$  of all lifts of  $\rho$  is the set of products  $\rho_\xi = \xi\tilde{\rho}$  with  $\xi \in C^1(K, \{A\})$ . Now

$$\sigma^r(\xi\tilde{\rho})(y) = (\xi\tilde{\rho})(x^{-r}yx^r) = \tau\xi(y)\tilde{\rho}(y),$$

so we can identify the dynamical system  $(\mathcal{P}, \sigma^r)$  with the Markov group  $(C^1(K, \{A\}), \tau)$ . The shift of finite type  $\tilde{O}$  is equal to  $\mathcal{P} \cup \sigma\mathcal{P} \cup \dots \cup \sigma^{r-1}\mathcal{P}$ . Clearly each irreducible component of  $(\mathcal{P}, \sigma^r)$  lies in a unique irreducible component of  $(\tilde{O}, \sigma)$ , and  $\tilde{O}$  is the disjoint union of these. The components of  $\tilde{O}$  are finite or uncountable according as the components of  $\mathcal{P}$  are finite or uncountable. ■

**Theorem 3.4.** Assume that  $K$  is a finitely presented  $\mathbb{Z}$ -group, and  $n = 2, 3$  or  $4$ . Then  $K$  has either finitely or uncountably many subgroups of index  $n$ .

**Proof.** It suffices to show that for these  $n$ ,  $\text{Hom}(K, S_n) = \Phi_{S_n}$  contains either finitely or uncountably many transitive representations. We may naturally regard  $\Phi_{S_2}$  and  $\Phi_{S_3}$  as subshifts of  $\Phi_{S_4}$  given by subgraphs of the graph describing  $\Phi_{S_4}$ .

Since  $S_2 \cong \mathbb{Z}_2$  is abelian,  $\Phi_{S_2}$  is a Markov group, and hence it is either finite or uncountable. Since all nontrivial elements are transitive, the theorem holds for  $n = 2$ .

If  $\Phi_{S_2}$  is uncountable, then the irreducible component containing the trivial representation is uncountable. In the graph describing  $\Phi_{S_2}$ , the edge corresponding to the trivial representation in  $\text{Hom}(B, S_2)$  begins and ends at the vertex  $v$  corresponding to the trivial representation in  $\text{Hom}(U, S_2)$ , and there must be another path  $p$  from  $v$  to itself corresponding to a nontrivial periodic representation  $\rho$ . Conjugating  $\rho$  with the transpositions (23) and (24) gives representations  $\rho'$  in  $\Phi_{S_3}$  and  $\rho''$  in  $\Phi_{S_4}$  that correspond to paths  $p'$  and  $p''$  from  $v$  to itself in the graphs describing those representation shifts. We have  $(12) \in \rho(K)$ , and so  $(13) \in \rho'(K)$  and  $(14) \in \rho''(K)$ . Freely concatenating  $p$  and  $p'$  gives uncountably many bi-infinite paths that correspond to transitive representations in  $\Phi_{S_3}$ ; concatenating the paths  $p, p'$  and  $p''$  gives uncountably many transitive representations in  $\Phi_{S_4}$ .

Suppose that  $\Phi_{S_2}$  is finite but  $\Phi_{S_3}$  is not. We have a short exact sequence  $\mathbb{Z}_3 \cong \langle (123) \rangle \rightarrow S_3 \rightarrow S_2$  that splits, giving an epimorphism  $\Phi_{S_3} \rightarrow \Phi_{S_2}$ . There must be a periodic representation  $\rho \in \Phi_{S_2}$  that has infinitely many lifts in  $\Phi_{S_3}$ ;  $\rho$  itself is one of them. Applying Lemma 3.3, we see that the irreducible component containing  $\rho$  in the lift of the orbit of  $\rho$  to  $\Phi_{S_3}$  is uncountable. Its graph contains a closed path  $p$  corresponding to  $\rho$  and another,  $\tilde{p}$ , corresponding to a lifting  $\tilde{\rho} = \xi\rho$  where  $\xi$  is a nontrivial element of  $C^1(K; \{A\})$ . Then either  $\tilde{\rho}(K)$  contains the 3-cycle (123), or  $\rho(K)$  contains (12) and  $\tilde{\rho}(K)$  contains (13) or (23). There are uncountably many bi-infinite paths in this component that contain both  $p$  and  $\tilde{p}$  and so correspond to transitive representations into  $S_3$ . Conjugation by the transposition (34) fixes  $\rho$  but sends  $\tilde{\rho}$  to a new periodic representation  $\bar{\rho} \in \Phi_{S_4}$ , which must therefore be in the same irreducible component of  $\Phi_{S_4}$ .

as  $\rho$  and  $\tilde{\rho}$ . By a similar argument, this component contains uncountably many transitive representations into  $S_4$ .

Finally, suppose that  $\Phi_{S_3}$  is finite. Applying Lemma 3.3 to the extension  $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \cong \langle (12)(34), (13)(24) \rangle \rightarrow S_4 \rightarrow S_3$  and to each of the periodic orbits of  $\Phi_{S_3}$ , we see that  $\Phi_{S_4}$  is a disjoint union of irreducible components, and thus has dense set of periodic points. Each component is finite or uncountable. If  $\Phi_{S_4}$  contains infinitely many transitive representations, then so does some irreducible component. We can find a closed path  $p$  in the graph of this component corresponding to a periodic transitive representation. Then uncountably many bi-infinite paths in this graph contain  $p$  and so correspond to transitive representations. ■

**Corollary 3.5.** Let  $K$  be a finitely presented  $\mathbb{Z}$ -group. If  $K$  contains infinitely many subgroups of index  $r = 2, 3$  or  $4$ , then  $K$  contains uncountably many subgroups of any index greater than or equal to  $r!$ .

**Proof.** Assume that  $K$  contains infinitely many subgroups of index  $r = 2, 3$  or  $4$ . Theorem 3.4 implies that there are in fact uncountably many. Consequently,  $K$  admits uncountably many transitive representations into  $S_r$ . It follows that among the kernels of the representations there are uncountably many subgroups of index no greater than  $r!$ . Theorem 1.2 of [SW99] implies that  $K$  contains uncountably many subgroups (not necessarily normal) of any index greater than or equal to  $r!$ . ■

It is not difficult to see that the conclusion of Theorem 3.4 holds for arbitrary groups when  $r = 2$ . However, it does not hold for arbitrary groups when  $r = 3, 4$ . Nor does it hold for  $\mathbb{Z}$ -groups when  $r = 5$ . The assertion for  $r = 3$  is established by the following example, a slight modification of an example generously provided by Jim Howie.

**Example 3.6.** Every index-3 subgroup  $H$  of a group  $G$  arises as the preimage  $\rho^{-1}(\langle (12) \rangle)$ , for some transitive representation  $\rho : K \rightarrow S_3$ . Moreover,  $H$  is normal if and only if the image of  $\rho$  is abelian.

Let  $G$  be the direct sum of countably many copies of  $S_3$ . Coordinate projections are transitive representations, yielding countably many subgroups of index 3. We will show that  $G$  has no other subgroups  $H$  of index 3.

Observe that  $H$  cannot be normal. If it were, then  $G/H \cong \mathbb{Z}/3$  would be a quotient of  $G/[G, G]$ . However,  $G/[G, G]$  is a direct sum of countably many copies of  $\mathbb{Z}/2$ , and any quotient has exponent 2. Hence  $H$  arises from a surjective representation  $\rho : G \rightarrow S_3$ . We can see that the only such representations are projections composed with automorphisms of  $S_3$ . For let  $a = (12), b = (123)$ , which generate  $S_3$ . We regard  $G$  as generated by  $a_i, b_i$ ,  $i \in \mathbb{N}$ , where  $a_i, b_i$  are generators of the  $i$ th factor of  $G$ . Since the image of  $\rho$  is not abelian, some  $b_j$  must be mapped to an element of order 3. The corresponding generator  $a_j$  must be sent to an element of order 2. Then no other generator  $a_i, b_i$ ,  $i \neq j$  can be mapped nontrivially, since such generators commute with  $a_j$  and  $b_j$  in the direct sum.

In a similar way, one can show that the direct sum of countably many copies of  $A_4$  contains countably but not uncountably many subgroups of index 4. Details are left to the reader.

Example 3.6 is notable in another respect. Using a trick of [St84] (see page 273), one can see that  $G$  is the kernel of homomorphism from a finitely presented group onto  $\mathbb{Z}^2$ . Hence the conclusion of Theorem 3.4 does not hold for finitely presented  $\mathbb{Z}^d$ -groups, defined in the obvious manner, when  $d$  is greater than 1.

We show that the conclusion of Theorem 3.4 does not hold when  $r = 5$  by modifying Example 3.2 of [SW99'], an example due to K.H. Kim and F. Roush.

**Example 3.7.** The alternating group  $A_5$  has presentation  $\langle a, b \mid a^2, b^3, (ab)^5 \rangle$ . Consider the HNN extension  $G = \langle x, B \mid x^{-1}ax = \phi(a), \forall a \in U \rangle$ , where

$$B = \langle a, b, a', b' \mid a^2, b^3, (ab)^5, a'^2, b'^3, (a'b')^5, [a, a'], [b, b'], [aa'^{-1}, b'] [bb'^{-1}, a'] \rangle,$$

a quotient of the free product of two copies of  $A_5$ ;  $U$  and  $V$  are the subgroups generated by  $a, b$  and  $a', b'$  respectively; and the homomorphism  $\phi : U \rightarrow V$  maps  $a \mapsto a', b \mapsto b'$ . As in §2, let  $K$  be the kernel of the epimorphism  $\chi : G \rightarrow \mathbb{Z}$  that sends  $x \mapsto 1$  and  $a, b, a', b' \mapsto 0$ . We will show that the representation shift  $\Phi_{S_5}$  is countable.

We construct a graph describing  $\Phi_{S_5}$  as in section 2. A vertex  $v \in \text{Hom}(U, S_5)$  is determined by  $v(a) = \alpha$ ,  $v(b) = \beta$ , so we can identify the vertex set with the set of pairs  $(\alpha, \beta) \in S_5 \times S_5$  with  $\alpha^2 = \beta^3 = (\alpha\beta)^5 = e$ . There is an edge  $\bar{\rho}$  from  $(\alpha, \beta)$  to  $(\alpha', \beta')$  if and only if the assignment  $\bar{\rho}(a) = \alpha$ ,  $\bar{\rho}(b) = \beta$ ,  $\bar{\rho}(a') = \alpha'$ ,  $\bar{\rho}(b') = \beta'$  defines an element of  $\text{Hom}(B, S_5)$ . Since  $A_5$  is a simple subgroup of index 2 in  $S_5$ , the subgroup generated by  $\alpha$  and  $\beta$  is trivial, cyclic of order 2 or  $A_5$ . The second case, in fact, cannot occur:  $\beta$  would have to be trivial, thereby forcing  $\alpha^2 = \alpha^5 = e$  and so  $\alpha = e$  as well.

Note that every vertex admits a self-loop (that is, an edge which begins and ends at the vertex) and an edge from it to the vertex  $(e, e)$ . As in [SW99'], we claim that there are no other edges in  $\Gamma$ . To see this, suppose that there exists an edge from  $(e, e)$  to some other vertex  $(\alpha, \beta)$ . From the presentation above, we see that  $\alpha$  and  $\beta$  must commute, and hence  $\alpha = \beta = e$ . Now suppose that there exists an edge from a vertex  $(\alpha, \beta)$  to another  $(\gamma, \delta)$ , neither of which is  $(e, e)$ . Again using the presentation we see that  $\alpha\gamma^{-1}$  and  $\beta\delta^{-1}$  commute with both  $\gamma$  and  $\delta$ . Since the center of  $A_5$  is trivial,  $\alpha = \gamma$  and  $\beta = \delta$  and hence the edge is merely a self-loop on  $(\alpha, \beta)$ .

It is clear that the graph we have described has countably many bi-infinite paths, and so the representation shift  $\Phi_{S_5}$  is countable. All of these representations except the trivial one are transitive. Hence  $K$  has countably many subgroups of index 5 and none of index 2, 3 or 4.

**Theorem 3.8.** Let  $K$  be a finitely presented  $\mathbb{Z}$ -group. Suppose some representation  $\rho$  of  $K$  onto  $S_2$  has infinitely many lifts to representations into  $S_3$ . Then  $K$  has uncountably many representations onto  $S_n$  for all  $n \geq 3$ .

**Proof.** The argument is similar to that of the next-to-last paragraph of the proof of Theorem 3.4. By Lemma 3.3, the component of  $\rho$  in the lifting of  $\rho$  to  $S_3$  is uncountable. The graph of this component contains closed paths  $p$  and  $\tilde{p}$  corresponding to  $\rho$  and a nontrivial periodic lifting  $\tilde{\rho}$ . The image  $\tilde{\rho}(K)$  must contain (13), (23) or (123), so paths that contain both  $p$  and  $\tilde{p}$  correspond to representations onto  $S_3$ .

Now fix  $n \geq 4$ . For  $4 \leq m \leq n$  we can obtain a periodic representation  $\rho^{(m)}$  of  $K$  onto the subgroup of  $S_n$  consisting of all permutations of  $\{1, 2, m\}$  by conjugating  $\tilde{\rho}$  by the 2-cycle  $(3m)$ . Since each of these conjugations fixes  $\rho$ , they all leave the component of  $\rho$  in the graph of  $\Phi_{S_n}$  invariant. Any path in this component that contains the closed paths corresponding to  $\rho$ ,  $\tilde{\rho}$  and each  $\rho^{(m)}$  must map onto  $S_n$ , and there are uncountably many such paths. ■



**Corollary 3.9.** Let  $K$  be the commutator subgroup of a knot group. If  $K$  has infinitely many representations into  $S_3$ , then  $K$  has uncountably many representations onto  $S_n$  for all  $n \geq 3$ , and hence uncountably many subgroups of every index  $n \geq 3$ .

**Proof.** As we noted at the end of Section 2, the Markov groups  $\Phi_{\mathbb{Z}_2}$  and  $\Phi_{\mathbb{Z}_3}$  are finite in this case. Hence there are infinitely many representations onto  $S_3$ , and infinitely many are lifts of a single representation onto  $S_2$ . ■

**4. Additional application to knot groups.** Assume that  $M \cong R^n/AR^m$  is a finitely generated module over a Noetherian ring  $R$ , where the presentation matrix  $A$  is an  $n \times m$  matrix with entries in  $R$ . By adjoining zero columns, we can assume that  $m \geq n$ . The elementary ideals  $E_i$  of  $A$  form a sequence of invariants of  $m$ . The ideal  $E_i$  is generated by the  $(n-i) \times (n-i)$  minors of the matrix  $A$ . When  $R$  is a factorial domain (for example,  $\mathbb{Z}[t, t^{-1}]$  or  $F[t, t^{-1}]$ , where  $F$  is a field), each  $E_i$  is contained in a unique minimal principal ideal; a generator, which is well defined up to multiplication by units in  $R$ , is denoted by  $\Delta_i(t)$ . If  $R = \mathbb{Z}[t, t^{-1}]$ , then  $\Delta_i$  is a polynomial, the  $i$ th *characteristic polynomial* of  $M$ , and we normalize it so that it has the form  $c_0 + c_1t + \cdots + c_d t^d$ , where  $c_0 \neq 0$ .

Here we are concerned only with the 0th characteristic polynomial. Note that when  $A$  is a square matrix,  $\Delta_0(t)$  is simply the determinant of  $A$ . Such is the case for any knot: If  $X = S^3 \setminus k$  is a knot complement, and  $\tilde{X}$  is its infinite cyclic cover, then  $H_1(\tilde{X}; \mathbb{Z})$  is a finitely generated  $\mathbb{Z}[t, t^{-1}]$ -module with square presentation matrix. The 0th characteristic polynomial is called the *Alexander polynomial* of  $k$ , denoted here by  $\Delta(t)$ . The Alexander polynomial  $\Delta(t)$  of any knot has even degree and satisfies  $\Delta(1) = \pm 1$ .

For any positive integer  $r$ , we can regard  $M$  as a finitely generated module over  $\mathbb{Z}[s, s^{-1}]$ , where  $s = t^r$ . We can obtain a presentation matrix  $A(C_r)$  from  $A$  by replacing each polynomial entry  $f(t)$  by  $f(C_r)$ , where  $C_r$  is the  $r \times r$  companion matrix of the polynomial  $t^r - s$ :

$$C_r = \begin{pmatrix} 0 & 0 & \cdots & 0 & s \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

**Lemma 4.1.** Assume that  $M \cong R^n/AR^n$  is a finitely generated  $R = \mathbb{Z}[t, t^{-1}]$ -module with 0th characteristic polynomial  $\Delta(t) = c_d \prod (t - \alpha_i)$ . Let  $s = t^r, r \geq 1$ . The 0th characteristic polynomial of  $M$  regarded as a  $\mathbb{Z}[s, s^{-1}]$ -module is  $\tilde{\Delta}(s) = c_d^r \prod (s - \alpha_i^r)$ .

**Proof.** Regard  $\mathbb{Z}[s, s^{-1}]$  as a subring of  $\mathbb{Z}[t, t^{-1}]$ , which in turn is a subring of  $\mathbb{C}[t, t^{-1}]$ . The matrix  $C_r$  is similar over  $\mathbb{C}[t, t^{-1}]$  to the diagonal matrix  $D = \text{Diag}(t, \zeta t, \dots, \zeta^{r-1} t)$ , where  $\zeta$  is a primitive  $r$ th root of unity. Consequently,  $\tilde{\Delta}(s) = \text{Det} A(D)$ , where as above  $A(D)$  denotes the matrix obtained from  $A$  by replacing each polynomial entry  $f(t)$  by  $f(D)$ . By Theorem 1 of [KSW99] the determinant is equal to  $\text{Det}[\Delta(D)]$ , which is  $\text{Det}[c_d \prod_i (D - \alpha_i I)] = c_d^r \prod_i \prod_j (\zeta^j t - \alpha_i)$ . The last product can be rewritten as  $c_d^r \prod_i \prod_j (t - \zeta^{-j} \alpha_i)$ , which is equal to  $c_d^r \prod_i (t^r - \alpha_i^r) = c_d^r \prod_i (s - \alpha_i^r)$ . ■

A polynomial  $f(t) \in \mathbb{Z}[t, t^{-1}]$  of degree  $d$  is *symmetric* (or *reciprocal*) if  $f(t^{-1}) \doteq f(t)$ . Here  $\doteq$  indicates equality up to multiplication by a unit  $\pm t^n$  in  $\mathbb{Z}[t, t^{-1}]$ . It is well known that the Alexander polynomial  $\Delta(t)$  of any knot is a symmetric polynomial of even degree (see [BS03], for example).

**Corollary 4.2.** Assume the hypotheses of Lemma 4.1. If, moreover,  $\Delta(t)$  is symmetric, then so is  $\tilde{\Delta}(s)$ .

**Proof.** A polynomial is symmetric if and only if its set of zeros (with multiplicities) is sent to itself by inversion. The desired conclusion follows immediately. ■

**Theorem 4.3.** Assume that  $K$  is the commutator subgroup of the group of a knot. Then (i) any representation from  $K$  onto  $S_2$  lifts to a representation onto  $S_3$ ; (ii) any representation from  $K$  onto  $\mathbb{Z}_3$  lifts to a representation onto  $A_4$ .

**Proof.** (i) Let  $\rho$  be a representation from  $K$  onto  $S_2$ . The (unique) epimorphism  $S_3 \rightarrow S_2$  fits into a short exact sequence  $A = \langle (123) \rangle \rightarrow S_3 \rightarrow S_2$  that splits, and hence a lifting  $\tilde{\rho}$  can be found. By Proposition 3.1, the complete set of liftings  $\rho_\xi$  corresponds bijectively to the group of twisted cocycles  $C^1(K, \{A\})$ . It is not difficult to see  $\rho_\xi$  fails to be surjective if and only if  $\xi$  is a coboundary. We must prove that  $H^1(K, \{A\})$  is nontrivial.

Our proof is topological. We construct a 2-complex  $X$  with fundamental group  $K$ , and invoke Theorem 4 of [Li95], an application of Shapiro's Lemma, to see that the problem of showing that  $H^1(K, \{A\})$  is nontrivial is equivalent to proving that the mod-3 first homology group of a certain cover of  $X$  has larger rank than then corresponding homology group of  $X$ . The fact that  $K$  is the commutator subgroup of a knot ensures that the cover satisfies a type of Poincaré duality, imposing conditions that are sufficient to complete the argument.

Let  $X$  be a CW complex with a single vertex and fundamental group  $K$ . For any prime  $p$ , the homology group  $H_1(X; \mathbb{Z}_p)$  is a finitely generated module over the ring  $\mathbb{Z}_p[t, t^{-1}]$  of Laurent polynomials with coefficients in  $\mathbb{Z}_p$ . A square matrix presenting the module can be found, and its determinant  $\Delta_0(H_1(X; \mathbb{Z}_p))$  is the Alexander polynomial  $\Delta(t)$  of the knot with coefficients reduced modulo  $p$ . Alternatively,  $H_1(X; \mathbb{Z}_p)$  can be viewed as a finite-dimensional vector space over  $\mathbb{Z}_p$ . Its dimension is equal to the mod  $p$  degree of  $\Delta(t)$ . All of the above statements hold as well using cohomology.

Let  $\pi : \tilde{X} \rightarrow X$  be the 2-fold cover corresponding to  $\rho$ . By Theorem 4 of [Li95],  $H^1(K, \{A\}) \cong H^1(\tilde{X}; \mathbb{Z}_3)/\pi^* H^1(X; \mathbb{Z}_3)$ . We must prove that the dimension of  $H^1(\tilde{X}; \mathbb{Z}_3)$  exceeds that of  $H^1(X; \mathbb{Z}_3)$ . The Universal Coefficient Theorem implies that the dimension of  $H^1(\tilde{X}; \mathbb{Z}_3)$  (resp.  $H^1(X; \mathbb{Z}_3)$ ) is equal to that of  $H_1(\tilde{X}; \mathbb{Z}_3)$  (resp.  $H_1(X; \mathbb{Z}_3)$ ). We will work with homology.

Any representation from  $K$  to a finite abelian group is periodic [SW99']. Let  $r$  be the period of  $\rho$ . Then  $H_1(\tilde{X}; \mathbb{Z})$  is a finitely generated module over  $\Lambda = \mathbb{Z}[s, s^{-1}]$ , where  $s = t^r$ . In fact  $\rho$  induces a homomorphism from the fundamental group of the  $r$ -fold cyclic cover  $X_r$  of the knot to  $S_2$ . (Details can be found in [SW99'].) We can regard  $\tilde{X}$  as an infinite cyclic cover of the induced 2-fold cover of  $X_r$ . It follows from Blanchfield duality that  $\Delta_0(H_1(\tilde{X}; \mathbb{Z}))$  is a symmetric polynomial ([Tu01], Corollary 14.7).

We regard  $H_1(X; \mathbb{Z})$  too as a  $\Lambda$ -module with 0th characteristic polynomial  $\tilde{\Delta}(s)$ . By Lemma 4.1 and Corollary 4.2,  $\tilde{\Delta}(s)$  has the same degree as  $\Delta(t)$  and is symmetric.

The CW complex  $\tilde{X}$  has a 0-skeleton  $\tilde{X}^0$  consisting of two vertices. It is convenient to work with the relative homology group  $H_1(\tilde{X}, \tilde{X}^0; \mathbb{Z})$ ; it fits into a short exact sequence

$$0 \rightarrow H_1(\tilde{X}; \mathbb{Z}) \rightarrow H_1(\tilde{X}, \tilde{X}^0; \mathbb{Z}) \rightarrow \Lambda/(s-1) \rightarrow 0,$$

from which it follows that

$$\Delta_0(H_1(\tilde{X}, \tilde{X}^0; \mathbb{Z})) \doteq \Delta_0(H_1(\tilde{X}; \mathbb{Z}))(s-1)$$

(see, for example, Lemma 7.2.7 [Ka96]). Consider then the chain complex

$$0 \rightarrow C_2(\tilde{X}, \tilde{X}^0; \mathbb{Z}) \xrightarrow{\partial} C_1(\tilde{X}, \tilde{X}^0; \mathbb{Z}) \rightarrow 0$$

for the pair  $(\tilde{X}, \tilde{X}^0)$ . The boundary  $\partial$  can be represented by a matrix of the form

$$T = \begin{pmatrix} A & B \\ B & A \end{pmatrix}.$$

Here  $A$  and  $B$  are square matrices of the same size. The first half of the columns of  $T$  correspond to edges of  $\tilde{X}$  that are lifts of edges of  $X$  beginning at a fixed vertex of  $\tilde{X}^0$ ; the remaining columns correspond to edges that are lifts beginning at the other vertex.

Row and column operations convert  $T$  into

$$T' = \begin{pmatrix} A - B & B \\ 0 & A + B \end{pmatrix}.$$

The matrix  $A + B$  is a relation matrix for  $H_1(X; \mathbb{Z})$ , and consequently its determinant is  $\tilde{\Delta}(s)$ . Since  $\Delta_0(H_1(\tilde{X}; \mathbb{Z}))$  and  $\tilde{\Delta}(s)$  are both symmetric polynomials, so is  $\det(A - B)$ . The latter factors as  $(s-1)g(s)$  for some (necessarily symmetric) polynomial  $g(s)$ , since  $s-1$  cannot divide  $\tilde{\Delta}(s)$ .

Clearly  $\det(A - B)$  is congruent modulo 2 to  $\det(A + B)$ . It follows that  $g(s)$  has odd degree. If  $g(s) \pmod{3}$  is nonzero, then it must have positive degree. Therefore, if  $\Delta_0(H_1(\tilde{X}; \mathbb{Z}_3))$  is nonzero, then its degree is larger than that of  $\Delta_0(H_1(X; \mathbb{Z}_3))$ . Equivalently, if  $H_1(\tilde{X}; \mathbb{Z}_3)$  has finite dimension, then its dimension is greater than that of  $H_1(X; \mathbb{Z}_3)$ .

The proof of (ii) follows a similar line of reasoning as (i). Let  $\rho$  be a representation of  $K$  onto  $\mathbb{Z}_3$ . The alternating group  $A_4$  maps onto  $\mathbb{Z}_3$  with kernel  $A = \langle (12)(34), (13)(24) \rangle \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ , and the resulting short exact sequence splits. Hence a lifting  $\tilde{\rho} : K \rightarrow A_4$  can be found. As in the proof of (i), Proposition 3.1 implies that the complete set of liftings  $\rho_\xi$  corresponds bijectively to the group of twisted cocycles  $C^1(K, \{A\})$ . Again  $\rho_\xi$  fails to be surjective if and only if  $\xi$  is a coboundary. We must prove that  $H^1(K, \{A\})$  is nontrivial.

Let  $p : \tilde{X} \rightarrow X$  be the 3-fold cover corresponding to  $\rho$ . By Theorem 4 of [Li95],  $H^1(K, \{A\}) \cong H^1(\tilde{X}; \mathbb{Z}_2)/p^*H^1(X; \mathbb{Z}_2)$ . Hence it suffices to show that the dimension of  $H_1(\tilde{X}; \mathbb{Z}_2)$  exceeds the dimension of  $H_1(X; \mathbb{Z}_2)$ .

Since the unique vertex of  $X$  is covered by 3 vertices in  $\tilde{X}$ , a short exact homology sequence similar to the one above shows that

$$\Delta_0(H^1(\tilde{X}, \tilde{X}^0; \mathbb{Z}_2)) \doteq \Delta_0(H^1(\tilde{X}; \mathbb{Z}_2))(s-1)^2.$$

The representation  $\rho : K \rightarrow \mathbb{Z}_3$  is necessarily periodic, say of period  $r$ . Let  $s = t^r$ . Then  $H_1(\tilde{X}, \tilde{X}^0; \mathbb{Z}_2)$  is a finitely generated  $\Lambda = \mathbb{Z}[s, s^{-1}]$ -module. We view  $H_1(X, X^0; \mathbb{Z}_2)$  likewise as a  $\Lambda$ -module, and again by Blanchfield Duality and Corollary 4.2 its 0th characteristic polynomial  $\tilde{\Delta}(s)$  is symmetric of even degree.

As in the proof of (i) we consider the chain complex

$$0 \rightarrow C_2(\tilde{X}, \tilde{X}^0; \mathbb{Z}) \xrightarrow{\partial} C_1(\tilde{X}, \tilde{X}^0; \mathbb{Z}) \rightarrow 0$$

for the pair  $(\tilde{X}, \tilde{X}^0)$ . The boundary operator  $\partial$  is represented by a matrix of the form

$$T = \begin{pmatrix} A & B & C \\ B & C & A \\ C & A & B \end{pmatrix},$$

where  $A, B$  and  $C$  are square matrices of the same size. The first third of the columns of  $T$  correspond to edges of  $\tilde{X}$  that are lifts of edges of  $X$  beginning at a fixed vertex of  $\tilde{X}^0$ ; the second third (resp. last third) correspond to edges that are lifts beginning at the second (resp. third) vertex.

The matrix  $T$  is similar in  $\mathbb{Z}[\zeta][s, s^{-1}]$  to

$$T' = \begin{pmatrix} A + B + C & 0 & 0 \\ 0 & A + \zeta B + \zeta^2 C & 0 \\ 0 & 0 & A + \zeta^2 B + \zeta C \end{pmatrix},$$

where  $\zeta$  is a primitive 3rd root of unity. Consequently,  $\Delta_0(H_1(\tilde{X}, \tilde{X}^0; \mathbb{Z}))$  is equal to  $\text{Det}(R') = \text{Det}(A + B + C)\text{Det}(A + \zeta B + \zeta^2 C)\text{Det}(A + \zeta^2 B + \zeta C)$ , which we write as  $\tilde{\Delta}(s)F(s)\bar{F}(s)$ . As before  $\tilde{\Delta}(s)$  is obtained from the Alexander polynomial of the knot, using Lemma 4.1;  $F(s)$  is a polynomial with coefficients in  $\mathbb{Z}[\zeta]$ , and  $\bar{F}(s)$  is the polynomial obtained from  $F(s)$  by replacing each coefficient by its conjugate.

As in the proof of (i),  $\Delta_0(H_1(\tilde{X}; \mathbb{Z}))$  is a symmetric polynomial, and hence so is  $\Delta_0(H_1(\tilde{X}, \tilde{X}^0; \mathbb{Z}))$ . Since  $\Delta_0(H_1(\tilde{X}, \tilde{X}^0; \mathbb{Z})) = \tilde{\Delta}(s)F(s)\bar{F}(s)$  and  $\tilde{\Delta}(s)$  are symmetric, so is  $F(s)\bar{F}(s)$ . By the Universal Coefficient Theorem,  $\Delta_0(H_1(\tilde{X}, \tilde{X}^0; \mathbb{Z}_2))$  is equal to  $\tilde{\Delta}(s)F(s)\bar{F}(s)$  with coefficients reduced modulo 2.

The product  $F(s)\bar{F}(s)$  has integer coefficients. Since the unique extension of the natural projection  $\mathbb{Z} \rightarrow \mathbb{Z}_3$  to  $\mathbb{Z}[\zeta]$  sends  $\zeta$  to 1, the product  $F(s)\bar{F}(s)$  is congruent modulo 3 to  $(\tilde{\Delta}(s))^2$ . After suitable normalization, we can write  $F(s) = c_d s^d + \dots + c_1 s + c_0$ , where each  $c_i \in \mathbb{Z}[\zeta]$ . The product  $F(s)\bar{F}(s)$  is equal to  $c_d \bar{c}_d s^{2d} + \dots + c_0 \bar{c}_0$ . If the degree of  $F(s)\bar{F}(s)$  decreases when coefficients are reduced modulo 3, then both  $c_d \bar{c}_d$  and  $c_0 \bar{c}_0$  must be divisible by 3. This implies that  $c_d \bar{c}_d$  and  $c_0 \bar{c}_0$  vanish under  $\mathbb{Z}[\zeta] \rightarrow \mathbb{Z}_3$ , and hence both  $c_d$  and  $c_0$  also vanish. An induction argument shows that the degrees of  $F(s)\bar{F}(s)$  and  $F(s)\bar{F}(s) \pmod{3}$  differ by a multiple of 4. But since  $\tilde{\Delta}(s)$  has even degree, the degree of  $(\tilde{\Delta}(s))^2$  is a multiple of 4. Hence the degree of  $F(s)\bar{F}(s)$  is a multiple of 4, and so the degree of  $F(s)$  is even.

Now consider  $F(s)\bar{F}(s)$  with coefficients reduced modulo 2. As before, if the reduction causes its degree to decrease, then both  $c_d \bar{c}_d$  and  $c_0 \bar{c}_0$  must be even. This implies that both vanish under the natural projection  $\mathbb{Z}[\zeta] \rightarrow \mathbb{Z}_2[\zeta]$ . Hence  $F(s)\bar{F}(s) \pmod{2}$  has degree divisible by 4. If it is zero, then  $H_1(\tilde{X}; \mathbb{Z}_2)$  is infinite, and so its dimension is larger than that of  $H_1(X; \mathbb{Z}_2)$ . If it is nonzero, then it contains a nontrivial factor other than  $(s-1)^2$ , and again the dimension of  $H_1(\tilde{X}; \mathbb{Z}_2)$  is greater than that of  $H_1(X; \mathbb{Z}_2)$ . ■

**Example 4.4.** It is easy to construct examples of general  $\mathbb{Z}$ -groups for which the conclusions of Theorem 4.3 do not hold. For example, consider the group  $S_2$  presented in the following way.

$$K = \langle a_j \mid a_j^2, a_{j+1} = a_j, \forall j \rangle.$$

Clearly  $K$  admits a (unique) homomorphism onto  $S_2$  that does not lift to  $S_3$ . A similar example can be constructed to show that the second conclusion of Theorem 4.3 does not hold for general finitely presented  $\mathbb{Z}$ -groups.

Nontrivial examples can also be constructed. Consider the  $\mathbb{Z}$ -group:

$$K = \langle a_j \mid a_{j+1} = a_j^3 \ \forall j \rangle.$$

This group admits a representation  $\rho$  onto  $S_2$ , mapping each generator  $a_j$  to the nontrivial element. The reader can verify that the matrix  $T$  in the proof of Theorem 4.3 is

$$\begin{pmatrix} s-2 & -1 \\ -1 & s-2 \end{pmatrix}.$$

The determinant modulo 3 is equal up to a multiplicative unit to  $s-1$ , and hence the polynomial  $g(s)$  is trivial. Consequently,  $\rho$  does not lift onto  $S_3$ .

For the second part of Theorem 4.3, consider the group

$$K = \langle a_j \mid a_{j+1} = a_j^2 \ \forall j \rangle$$

of Example 2.1. This is the commutator subgroup of a 2-knot group. It admits a representation onto  $\mathbb{Z}_3$  mapping each  $a_{2j}$  to 2 and each  $a_{2j+1}$  to 1. The reader can verify that the matrix  $T$  in the proof of Theorem 4.3 is

$$\begin{pmatrix} 1 & -1 & 1 & 0 & 0 & 0 \\ -s & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & -s & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & -s & 1 \end{pmatrix}.$$

The determinant modulo 2 is equal up to a multiplicative unit to  $(s-1)^2$ , and hence  $F(s)\bar{F}(s)$  is trivial. Consequently,  $\rho$  does not lift onto  $A_4$ .

**5. Conclusion.** The obstruction theory used here has provided new insight into the structure of representation shifts. However, questions remain.

**Conjecture 5.1.** Dichotomy holds for commutator subgroups of knot groups. That is, for any finite target group  $\Sigma$ , the shift  $\Phi_\Sigma$  is either finite or uncountable.

A more subtle, dynamical conjecture which would imply Conjecture 5.1 is the following.

**Conjecture 5.2.** For any finite group  $\Sigma$ , periodic points are dense in every representation shift  $\Phi_\Sigma$  of a knot commutator subgroup.

A classical result of Perko [P76] says that every representation of a knot group onto  $S_3$  lifts onto  $S_4$ .

**Conjecture 5.3.** Let  $K$  be the commutator subgroup of a knot  $k$ . Every representation of  $K$  onto  $S_3$  lifts onto  $S_4$ .

Conjecture 5.3 holds for fibered knots, since in such a case  $K$  is a nonabelian free group. The general conjecture is equivalent to the assertion that for any surjection  $\rho : K \rightarrow S_3$ , the rank of the mod-2 first

homology group of the associated 3-fold dihedral cover  $\tilde{Y}$  of the infinite cyclic cover  $Y$  of the knot exceeds that of  $Y$ .

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